p-ADIC MULTIDIMENSIONAL WAVELETS AND THEIR APPLICATION TO p-ADIC PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. In this paper we study some problems related with the theory of multidimensional p-adic wavelets in connection with the theory of multidimensional p-adic pseudo-differential operators (in the p-adic Lizorkin space). We introduce a new class of n-dimensional p-adic compactly supported wavelets. In one-dimensional case this class includes the Kozyrev p-adic wavelets. These wavelets (and their Fourier transforms) form an orthonormal complete basis in $\mathcal{L}^2(\mathbb{Q}_p^n)$. A criterion for a multidimensional p-adic wavelet to be an eigenfunction for a pseudo-differential operator is derived. We prove that these wavelets are eigenfunctions of the Taibleson fractional operator. Since many p-adic models use pseudo-differential operators (fractional operator), these results can be intensively used in applications. Moreover, p-adic wavelets are used to construct solutions of linear and *semi-linear* pseudo-differential equations.

1. Introduction

There are a lot of papers where different applications of p-adic analysis to physical problems, stochastics, cognitive sciences and psychology are studied [6]– [10], [13]– [19], [30]– [32] (see also the references therein).

The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if an arbitrary rational number $x \neq 0$ is represented as $x = p^{\gamma}\frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$, and m and n are not divisible by p, then $|x|_p = p^{-\gamma}$. This norm in $\mathbb{Q}_p$ satisfies the strong triangle inequality $|x+y|_p \le \max(|x|_p, |y|_p)$.

It is known that for the p-adic analysis related to the mapping $\mathbb{Q}_p \to \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, the operation of partial differentiation is *not defined*, and as a result, large number of models connected with p-adic

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differential equations use pseudo-differential operators and the theory of p-adic distributions (generalized functions) (see the above mentioned papers and books). In particular, fractional operators $D^{\alpha} = f_{-\alpha}*$ are extensively used, where f_{α} is the p-adic *Riesz kernel*, $*$ is a convolution. However, in general, $D^{\alpha}\varphi \notin \mathcal{D}(\mathbb{Q}_p^n)$ for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, where $\mathcal{D}(\mathbb{Q}_p^n)$ is the space of test functions. Consequently, the operation $D^{\alpha}f$ is well defined only for some distributions $f \in \mathcal{D}'(\mathbb{Q}_p^n)$. For example, D^{-1} is *defined* only on the test functions such that $\int_{\mathbb{Q}_p} \varphi(x)\,dx = 0$ [30, IX.2].

We recall that similar problems arise for the "$\mathbb{C}$-case" fractional operators (where all functions and distributions are complex or real valued defined on spaces with real or complex coordinates): in general, the Schwartzian test function space $\mathcal{S}(\mathbb{R}^n)$ *is not invariant* under fractional operators [26], [27]. To solve this problem, in the excellent papers of P. I. Lizorkin [24], [25] a new type spaces *invariant* under fractional operators were introduced (see also [26], [27]).

Taking into account the problems mentioned above, in [3], the p-adic Lizorkin spaces of test functions and distributions were introduced, and in [3], [4] a class of pseudo-differential operators (including the Taibleson fractional operator) defined on them was constructed. The Lizorkin spaces are *invariant* under our pseudo-differential operators, and consequently, these spaces are their "natural" definition domains and can play a key role in considerations related to the fractional operators problems.

Recall that for the one-dimensional case the orthonormal complete basis of eigenfunctions (5.5) of the Vladimirov operator D^{α} was constructed by S. V. Kozyrev [20]. The eigenfunctions (5.5) are p-adic compactly supported wavelets. Further development and generalization of the theory of such type wavelets can be found in the papers by S. V. Kozyrev [21], [22], A. Yu. Khrennikov, and S. V. Kozyrev [16], [17], J. J. Benedetto, and R. L. Benedetto [8], and R. L. Benedetto [9].

It is typical that such type p-adic compactly supported wavelets are eigenfunctions of p-adic pseudo-differential operators. Moreover, these wavelets satisfy the condition $\int_{\mathbb{Q}_p} \varphi(x)\,dx = 0$ (see [20]), and, in view of Lemma 3.1, belong to the Lizorkin space $\Phi(\mathbb{Q}_p)$. In [3], there waqs derived the necessary and sufficient condition for multidimensional p-adic pseudo-differential operators to have such type multidimensional wavelets as eigenfunctions. Thus the wavelets theory play a key role in p-adic analysis.

Contents of the paper. In this paper problems related with the theory of multidimensional p-adic pseudo-differential operators and the theory of multidimensional p-adic wavelets are studied. Here the results of our paper [3] are intensively used.

In Sec. 2, we recall some facts from the p-adic theory of distributions [12], [28], [29], [30]. In Sec. 3, some facts from the theory of the p-adic Lizorkin

spaces [3] are recalled. In Sec. 4, we recall some facts on the multidimensional pseudo-differential operators defined in the Lizorkin space of distributions $\mathcal{D}'(\mathbb{Q}_p^n)$. The fractional Taibleson operator [28, §2], [29, III.4.] is among them. The Lizorkin spaces are *invariant* under our pseudo-differential operators. It is appropriate to mention here that the class of our operators includes the pseudo-differential operators studied in [19], [33], [34].

In Sec. 5, a *new type of p-adic compactly supported wavelets* (in one- dimensional (5.3) and multidimensional (5.17) cases) are introduced. These wavelets belong to the Lizorkin space of test functions. The Kozyrev one-dimensional wavelets [20] (see (5.5)) is a particular case of our one-dimensional wavelets (5.3). The *scaling function* of wavelets (5.3) is a characteristic function of the unit disc. The *two-scale equation* (5.7) for these wavelets is presented. However, in this paper the multiresolution analysis is not considered. The one-dimensional wavelets (5.3) and multidimensional wavelets (5.17) form orthonormal complete bases in $\mathcal{L}^2(\mathbb{Q}_p)$ and $\mathcal{L}^2(\mathbb{Q}_p^n)$, respectively (see Theorems 5.1, 5.2). Their Fourier transforms also form orthonormal complete bases in $\mathcal{L}^2(\mathbb{Q}_p)$ and $\mathcal{L}^2(\mathbb{Q}_p^n)$, respectively (see Corollary 5.1, 5.2).

In Sec. 6, the spectral theory of our pseudo-differential operators is constructed. By Theorem 6.1 the criterion (6.1) for multidimensional p-adic pseudo-differential operators (3.2) to have multidimensional wavelets (5.17) as eigenfunctions is derived. In particular, the multidimensional wavelets (5.17) are eigenfunctions of the Taibleson fractional operator (see (6.6)).

Since many p-adic models use pseudo-differential operators, in particular, fractional operator, these results on p-adic wavelets can be intensively used in applications. Moreover, p-adic wavelets can be used to construct solutions of linear and *semi-linear* pseudo-differential equations [5], [18], [23].

2. p-Adic distributions

Now we recall some facts from the theory of p-adic distributions (generalized functions). We shall systematically use the notations and results from [30]. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the sets of positive integers, integers, complex numbers, respectively. Denote by $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ the multiplicative group of the field $\mathbb{Q}_p$. The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of points $x = (x_1, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2 \dots, n$, $n \ge 2$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p = \max_{1 \le j \le n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \tag{2.1}$$

Denote by $B_\gamma^n(a) = \{x : |x - a|_p \le p^\gamma\}$ the ball of radius p^γ with the center at a point $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$ and by $S_\gamma^n(a) = \{x : |x - a|_p = p^\gamma\} = B_\gamma^n(a) \setminus B_{\gamma-1}^n(a)$ its boundary (sphere), $\gamma \in \mathbb{Z}$. For $a = 0$ we set $B_\gamma^n(0) = B_\gamma^n$ and $S_\gamma^n(0) = S_\gamma^n$. For the case $n = 1$ we will omit the upper index n. Here

$$B_\gamma^n(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n), \tag{2.2}$$

where $B_\gamma(a_j)=\{x_j : |x_j-a_j|_p\le p^\gamma\}$ is a disc of radius p^γ with the center at a point $a_j\in\mathbb{Q}_p$, $j=1,2\dots,n$. Any two balls in $\mathbb{Q}_p^n$ either are disjoint or one contains the other. Every point of the ball is its center.

According to [30, I.3,Examples 1,2.], the disc B_γ is represented by the sum of $p^{\gamma-\gamma'}$ *disjoint* discs $B_{\gamma'}(a)$, $\gamma'<\gamma$:

$$B_\gamma=B_{\gamma'}\cup\cup_a B_{\gamma'}(a), \tag{2.3}$$

where $a=0$ and $a=a_{-r}p^{-r}+a_{-r+1}p^{-r+1}+\cdots+a_{-\gamma'-1}p^{-\gamma'-1}$ are the centers of the discs $B_{\gamma'}(a)$, $r=\gamma,\gamma-1,\gamma-2,\dots,\gamma'+1$, $0\le a_j\le p-1$, $a_{-r}\ne 0$.

In particular, the disc B_0 is represented by the sum of p *disjoint* discs

$$B_0=B_{-1}\cup\cup_{r=1}^{p-1}B_{-1}(r), \tag{2.4}$$

where $B_{-1}(r)=\{x\in S_0 : x_0=r\}=r+p\mathbb{Z}_p$, $r=1,\dots,p-1$; $B_{-1}=\{|x|_p\le p^{-1}\}=p\mathbb{Z}_p$; and $S_0=\{|x|_p=1\}=\cup_{r=1}^{p-1}B_{-1}(r)$. Here all the discs are disjoint. We call covering (2.3), (2.4) the *canonical covering* of the disc B_0.

A complex-valued function f defined on $\mathbb{Q}_p^n$ is called *locally-constant* if for any $x\in\mathbb{Q}_p^n$ there exists an integer $l(x)\in\mathbb{Z}$ such that

$$f(x+y)=f(x),\quad y\in B^n_{l(x)}.$$

Let $\mathcal{E}(\mathbb{Q}_p^n)$ and $\mathcal{D}(\mathbb{Q}_p^n)$ be the linear spaces of locally-constant $\mathbb{C}$-valued functions on $\mathbb{Q}_p^n$ and locally-constant $\mathbb{C}$-valued functions with compact supports (so-called test functions), respectively; $\mathcal{D}(\mathbb{Q}_p)$, $\mathcal{E}(\mathbb{Q}_p)$ [30, VI.1.,2.]. If $\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$, according to Lemma 1 from [30, VI.1.], there exists $l\in\mathbb{Z}$, such that

$$\varphi(x+y)=\varphi(x),\quad y\in B^n_l,\quad x\in\mathbb{Q}_p^n.$$

The largest of such numbers $l=l(\varphi)$ is called the *parameter of constancy* of the function φ. Let us denote by $\mathcal{D}^l_N(\mathbb{Q}_p^n)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^n)$ having supports in the ball B^n_N and with parameters of constancy $\ge l$ [30, VI.2.]. Denote by $\mathcal{D}'(\mathbb{Q}_p^n)$ the set of all linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ [30, VI.3.].

Let us introduce in $\mathcal{D}(\mathbb{Q}_p^n)$ a *canonical δ-sequence* $\delta_k(x)=p^{nk}\Omega(p^k|x|_p)$, and a *canonical* 1*-sequence* $\Delta_k(x)=\Omega(p^{-k}|x|_p)$, $k\in\mathbb{Z}$, $x\in\mathbb{Q}_p^n$, where

$$\Omega(t)=\begin{cases}1, & 0\le t\le 1,\\ 0, & t>1.\end{cases} \tag{2.5}$$

Here $\Delta_k(x)$ is the characteristic function of the ball B^n_k. It is clear [30, VI.3., VII.1.] that $\delta_k\to\delta$, $k\to\infty$ in $\mathcal{D}'(\mathbb{Q}_p^n)$ and $\Delta_k\to 1$, $k\to\infty$ in $\mathcal{E}(\mathbb{Q}_p^n)$.

The Fourier transform of $\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$ is defined by the formula

$$F[\varphi](\xi)=\int_{\mathbb{Q}_p^n}\chi_p(\xi\cdot x)\varphi(x)\,d^nx,\quad \xi\in\mathbb{Q}_p^n,$$

where $\chi_p(\xi\cdot x)=e^{2\pi i\sum_{j=1}^n\{\xi_jx_j\}_p}$; $\xi\cdot x$ is the scalar product of vectors; the function $\chi_p(\xi_jx_j)=e^{2\pi i\{\xi_jx_j\}_p}$ for every fixed $\xi_j\in\mathbb{Q}_p$ is an additive character

of the field $\mathbb{Q}_p$, $j=1,\dots,n$; $\{x\}_p$ is the *fractional part* of a number $x\in\mathbb{Q}_p$ which is defined as follows

$$\{x\}_p=\begin{cases} 0, \quad \text{if} \quad \gamma(x)\ge 0 \quad \text{or} \quad x=0, \\ p^{\gamma}(x_0+x_1p+x_2p^2+\cdots+x_{|\gamma|-1}p^{|\gamma|-1}), \quad \text{if} \quad \gamma(x)<0. \end{cases} \tag{2.6}$$

The Fourier transform is a linear isomorphism $\mathcal{D}(\mathbb{Q}_p^n)$ into $\mathcal{D}(\mathbb{Q}_p^n)$. Moreover, according to [28, Lemma A.], [29, III,(3.2)], [30, VII.2.],

$$\varphi(x)\in\mathcal{D}^l_N(\mathbb{Q}_p^n) \quad \text{iff} \quad F\big[\varphi(x)\big](\xi)\in\mathcal{D}^{-N}_{-l}(\mathbb{Q}_p^n). \tag{2.7}$$

We define the Fourier transform $F[f]$ of a distribution $f\in\mathcal{D}'(\mathbb{Q}_p^n)$ by the relation [30, VII.3.]:

$$\langle F[f],\varphi\rangle=\langle f,F[\varphi]\rangle, \quad \forall\,\varphi\in\mathcal{D}(\mathbb{Q}_p^n). \tag{2.8}$$

Let A be a matrix and $b\in\mathbb{Q}_p^n$. Then for a distribution $f\in\mathcal{D}'(\mathbb{Q}_p^n)$ the following relation holds [30, VII,(3.3)]:

$$F[f(Ax+b)](\xi)=|\det A|_p^{-1}\chi_p\big(-A^{-1}b\cdot\xi\big)F[f(x)]\big(A^{-1}\xi\big), \tag{2.9}$$

where $\det A\ne 0$. According to [30, IV,(3.1)],

$$F[\Delta_k](x)=\delta_k(x), \quad k\in\mathbb{Z}, \qquad x\in\mathbb{Q}_p^n. \tag{2.10}$$

In particular, $F[\Omega(|\xi|_p)](x)=\Omega(|x|_p)$.

The convolution $f*g$ for distributions $f,g\in\mathcal{D}'(\mathbb{Q}_p^n)$ is defined (see [30, VII.1.]) as

$$\langle f*g,\varphi\rangle=\lim_{k\to\infty}\langle f(x)\times g(y),\Delta_k(x)\varphi(x+y)\rangle \tag{2.11}$$

if the limit exists for all $\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$, where $f(x)\times g(y)$ is the direct product of distributions. If for distributions $f,g\in\mathcal{D}'(\mathbb{Q}_p^n)$ the convolution $f*g$ exists then [30, VII,(5.4)]

$$F[f*g]=F[f]F[g]. \tag{2.12}$$

It is well known (see, e.g., [30, III.2.]) that any *multiplicative character* π of the field $\mathbb{Q}_p$ can be represented as

$$\pi(x)\stackrel{def}{=}\pi_\alpha(x)=|x|_p^{\alpha-1}\pi_1(x), \quad x\in\mathbb{Q}_p,$$

where $\pi(p)=p^{1-\alpha}$ and $\pi_1(x)$ is a *normed multiplicative character* such that $\pi_1(x)=\pi_1(|x|_px)$, $\pi_1(p)=\pi_1(1)=1$, $|\pi_1(x)|=1$. We denote $\pi_0=|x|_p^{-1}$.

Definition 2.1. Let π_α be a multiplicative character of the field $\mathbb{Q}_p$.

(a) According to [1], [2], a distribution $f_m\in\mathcal{D}'(\mathbb{Q}_p)$ is said to be *associated homogeneous* (*in the wide sense*) of degree π_α and order m, $m\in\mathbb{N}\cup\{0\}$, if

$$\Big\langle f_m,\varphi\Big(\frac{x}{t}\Big)\Big\rangle=\pi_\alpha(t)|t|_p\langle f_m,\varphi\rangle+\sum_{j=1}^{m}\pi_\alpha(t)|t|_p\log_p^j|t|_p\langle f_{m-j},\varphi\rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ and $t \in \mathbb{Q}_p^*$, where $f_{m-j} \in \mathcal{D}'(\mathbb{Q}_p)$ is an associated homogeneous distribution of degree π_α and order $m-j$, $j = 1, 2, \dots, m$, i.e.,

$$f_m(tx) = \pi_\alpha(t) f_m(x) + \sum_{j=1}^{m} \pi_\alpha(t) \log_p^j |t|_p f_{m-j}(x), \quad t \in \mathbb{Q}_p^*.$$

If $m = 0$ we set that the above sum is empty.

(b) We say that a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ is *associated homogeneous* (*in the wide sense*) of degree π_α and order m, $m \in \mathbb{N} \cup \{0\}$, if for all $t \in \mathbb{Q}_p^*$ we have

$$f_m(tx) = f_m(tx_1, \dots, tx_n) = \pi_\alpha(t) f_m(x) + \sum_{j=1}^{m} \pi_\alpha(t) \log_p^j |t|_p f_{m-j}(x),$$

where $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, $f_{m-j} \in \mathcal{D}'(\mathbb{Q}_p^n)$ is an associated homogeneous distribution of degree π_α and order $m-j$, $j = 1, 2, \dots, m$. An *associated homogeneous* (*in the wide sense*) distribution of degree $\pi_\alpha(t) = |t|_p^{\alpha-1}$ and order m is called *associated homogeneous* of degree $\alpha - 1$ and order m.

(c) An associated homogeneous distribution (in the wide sense) of order $m = 1$ is called *associated homogeneous* distribution (see [11] and [1], [2]).

(d) An associated homogeneous distribution of degree π_α and order $m = 0$ is called *homogeneous* distribution of degree π_α, i.e.,

$$f_0(tx) = f_0(tx_1, \dots, tx_n) = \pi_\alpha(t) f_0(x), \quad x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

(for one-dimensional case see [12, Ch.II,§2.3.], [30, VIII.1.]).

The multidimensional homogeneous distribution $|x|_p^{\alpha-n} \in \mathcal{D}'(\mathbb{Q}_p^n)$ of degree $\alpha - n$ is constructed as follows. If $Re\,\alpha > 0$ then the function $|x|_p^{\alpha-n}$ generates a regular functional

$$\langle |x|_p^{\alpha-n}, \varphi \rangle = \int_{\mathbb{Q}_p^n} |x|_p^{\alpha-n} \varphi(x)\, d^n x, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{Q}_p^n), \tag{2.13}$$

where $|x|_p$, $x \in \mathbb{Q}_p^n$ is given by (2.1). If $Re\,\alpha \le 0$ this distribution is defined by means of analytic continuation [28, (*)], [29, III,(4.3)], [30, VIII,(4.2)]:

$$\langle |x|_p^{\alpha-n}, \varphi \rangle = \int_{B_0^n} |x|_p^{\alpha-n} \big(\varphi(x) - \varphi(0)\big)\, d^n x$$

$$+ \int_{\mathbb{Q}_p^n \setminus B_0^n} |x|_p^{\alpha-n} \varphi(x)\, d^n x + \varphi(0) \frac{1 - p^{-n}}{1 - p^{-\alpha}}, \tag{2.14}$$

for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, $\alpha \ne \mu_j = \frac{2\pi i}{\ln p} j$, $j \in \mathbb{Z}$. The distribution $|x|_p^{\alpha-n}$ is an entire function of the complex variable α everywhere except the points μ_j, $j \in \mathbb{Z}$, where it has simple poles with residues $\frac{1-p^{-n}}{\log p} \delta(x)$.

Similarly to the one-dimensional case [1], [2], one can construct the distribution $P(\frac{1}{|x|_p^n})$ called the principal value of the function $\frac{1}{|x|_p^n}$, $x \in \mathbb{Q}_p^n$:

$$\Big\langle P\Big(\frac{1}{|x|_p^n}\Big), \varphi\Big\rangle = \int_{B_0^n} \frac{\varphi(x)-\varphi(0)}{|x|_p^n}\, d^n x + \int_{\mathbb{Q}_p^n \setminus B_0^n} \frac{\varphi(x)}{|x|_p^n}\, d^n x, \tag{2.15}$$

for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. It is easy to show that this distribution is *associated homogeneous* of degree $-n$ and order 1 (see [1], [2]).

The Fourier transform of $|x|_p^{\alpha-n}$ is given by the formula from [28, Theorem 2.], [29, III,Theorem (4.5)], [30, VIII,(4.3)]

$$F[|x|_p^{\alpha-n}] = \Gamma_p^{(n)}(\alpha)|\xi|_p^{-\alpha}, \quad \alpha \neq 0,\, n \tag{2.16}$$

where the n-dimensional Γ-*function* $\Gamma_p^{(n)}(\alpha)$ is given by the following formulas (see [28, Theorem 1.], [29, III,Theorem (4.2)], [30, VIII,(4.4)]):

$$\Gamma_p^{(n)}(\alpha) \stackrel{def}{=} \lim_{k\to\infty} \int_{p^{-k}\le|x|_p\le p^k} |x|_p^{\alpha-n}\chi_p(u\cdot x)\, d^n x$$

$$= \int_{\mathbb{Q}_p^n} |x|_p^{\alpha-n}\chi_p(x_1)\, d^n x = \frac{1-p^{\alpha-n}}{1-p^{-\alpha}} \tag{2.17}$$

where $|u|_p = 1$, and the last integrals in the right-hand side of (2.17) are defined by means of analytic continuation with respect to the parameter α. Here $\Gamma_p^{(1)}(\alpha) = \Gamma_p(\alpha) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\chi_p(x)\, dx = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}$.

3. The p-adic Lizorkin spaces

Let us introduce the p-adic *Lizorkin space of test functions*

$$\Phi(\mathbb{Q}_p^n) = \{\phi : \phi = F[\psi],\, \psi \in \Psi(\mathbb{Q}_p^n)\},$$

where

$$\Psi(\mathbb{Q}_p^n) = \{\psi(\xi) \in \mathcal{D}(\mathbb{Q}_p^n) : \psi(0) = 0\}.$$

Here $\Psi(\mathbb{Q}_p^n), \Phi(\mathbb{Q}_p^n) \subset \mathcal{D}(\mathbb{Q}_p^n)$. The space $\Phi(\mathbb{Q}_p^n)$ can be equipped with the topology of the space $\mathcal{D}(\mathbb{Q}_p^n)$ which makes $\Phi(\mathbb{Q}_p^n)$ a complete space.

In view of (2.7), the following lemma holds.

Lemma 3.1. ([3], [4]) (a) $\phi \in \Phi(\mathbb{Q}_p^n)$ *iff* $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$ *and*

$$\int_{\mathbb{Q}_p^n} \phi(x)\, d^n x = 0. \tag{3.1}$$

(b) $\phi \in \mathcal{D}_N^l(\mathbb{Q}_p^n) \cap \Phi(\mathbb{Q}_p^n)$, *i.e.,* $\int_{B_N^n} \phi(x)\, d^n x = 0$, *iff* $\psi = F^{-1}[\phi] \in \mathcal{D}_{-l}^{-N}(\mathbb{Q}_p^n) \cap \Psi(\mathbb{Q}_p^n)$, *i.e.,* $\psi(\xi) = 0$, $\xi \in B_{-N}^n$.

In fact, for $n = 1$, this lemma was proved in [30, IX.2.]. Unlike the classical Lizorkin space, any function $\psi(\xi) \in \Phi(\mathbb{Q}_p^n)$ is equal to zero not only at $\xi = 0$ but in a ball $B^n \ni 0$, as well.

Let $\Phi'(\mathbb{Q}_p^n)$ denote the topological dual of the space $\Phi(\mathbb{Q}_p^n)$. We call it the p-adic *Lizorkin space of distributions*.

By $\Psi^\perp$ and $\Phi^\perp$ we denote the subspaces of functionals in $\mathcal{D}'(\mathbb{Q}_p^n)$ orthogonal to $\Psi(\mathbb{Q}_p^n)$ and $\Phi(\mathbb{Q}_p^n)$, respectively. Thus $\Psi^\perp = \{f \in \mathcal{D}'(\mathbb{Q}_p^n) : f = C\delta,\, C \in \mathbb{C}\}$ and $\Phi^\perp = \{f \in \mathcal{D}'(\mathbb{Q}_p^n) : f = C,\, C \in \mathbb{C}\}$.

Proposition 3.1. ([3])

$$\Phi'(\mathbb{Q}_p^n) = \mathcal{D}'(\mathbb{Q}_p^n)/\Phi^\perp, \qquad \Psi'(\mathbb{Q}_p^n) = \mathcal{D}'(\mathbb{Q}_p^n)/\Psi^\perp.$$

The space $\Phi'(\mathbb{Q}_p^n)$ can be obtained from $\mathcal{D}'(\mathbb{Q}_p^n)$ by "sifting out" constants. Thus two distributions in $\mathcal{D}'(\mathbb{Q}_p^n)$ differing by a constant are indistinguishable as elements of $\Phi'(\mathbb{Q}_p^n)$.

Similarly to (2.8), we define the Fourier transforms of distributions $f \in \Phi'_\times(\mathbb{Q}_p^n)$ and $g \in \Psi'_\times(\mathbb{Q}_p^n)$ by the relations:

$$\begin{aligned} \langle F[f], \psi\rangle &= \langle f, F[\psi]\rangle, \qquad \forall\, \psi \in \Psi(\mathbb{Q}_p^n), \\ \langle F[g], \phi\rangle &= \langle g, F[\phi]\rangle, \qquad \forall\, \phi \in \Phi(\mathbb{Q}_p^n). \end{aligned} \tag{3.2}$$

By definition, $F[\Phi(\mathbb{Q}_p^n)] = \Psi(\mathbb{Q}_p^n)$ and $F[\Psi(\mathbb{Q}_p^n)] = \Phi(\mathbb{Q}_p^n)$, i.e., (3.2) give well defined objects.

4. Pseudo-differential operators in the Lizorkin space

4.1. **Pseudo-differential operators.** Consider a class of pseudo-differential operators in the Lizorkin space of the test functions $\Phi(\mathbb{Q}_p^n)$

$$\begin{aligned} (A\phi)(x) &= F^{-1}\big[\mathcal{A}(\xi)\, F[\phi](\xi)\big](x) \\ &= \int_{\mathbb{Q}_p^n}\int_{\mathbb{Q}_p^n} \chi_p\big((y-x)\cdot\xi\big)\mathcal{A}(\xi)\phi(y)\, d^n\xi\, d^n y, \quad \phi \in \Phi(\mathbb{Q}_p^n) \end{aligned} \tag{4.1}$$

with symbols $\mathcal{A}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$.

Lemma 4.1. *The Lizorkin space $\Phi(\mathbb{Q}_p^n)$ is invariant under the pseudo- differential operators* (4.1)*. Moreover, $A(\Phi(\mathbb{Q}_p^n)) = \Phi(\mathbb{Q}_p^n)$.*

Proof. In view of (2.7) and results of Sec. 3, functions $F[\phi](\xi)$ and $\mathcal{A}(\xi)F[\phi](\xi)$ belong to $\Psi(\mathbb{Q}_p^n)$, and, consequently, $(A\phi)(x) \in \Phi(\mathbb{Q}_p^n)$, i.e., $A(\Phi(\mathbb{Q}_p^n)) \subset \Phi(\mathbb{Q}_p^n)$. Thus the pseudo-differential operators (4.1) are well defined, and the Lizorkin space $\Phi(\mathbb{Q}_p^n)$ is invariant under them. Moreover, any function from $\Psi(\mathbb{Q}_p^n)$ can be represented as $\psi(\xi) = \mathcal{A}(\xi)\psi_1(\xi)$, $\psi_1 \in \Psi(\mathbb{Q}_p^n)$. This implies that $A(\Phi(\mathbb{Q}_p^n)) = \Phi(\mathbb{Q}_p^n)$. $\square$

If we define a conjugate pseudo-differential operator A^T as

$$(A^T\phi)(x) = F^{-1}[\mathcal{A}(-\xi)F[\phi](\xi)](x) = \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\mathcal{A}(-\xi)F[\phi](\xi)\, d^n\xi \tag{4.2}$$

then one can define the operator A in the Lizorkin space of distributions: for $f \in \Phi'(\mathbb{Q}_p^n)$ we have

$$\langle Af, \phi\rangle = \langle f, A^T \phi\rangle, \qquad \forall\, \phi \in \Phi(\mathbb{Q}_p^n). \tag{4.3}$$

It is clear that

$$Af = F^{-1}[\mathcal{A}\, F[f]] \in \Phi'(\mathbb{Q}_p^n), \tag{4.4}$$

i.e., the Lizorkin space of distributions $\Phi'(\mathbb{Q}_p^n)$ is invariant under pseudo-differential operators A. Moreover, in view of Lemma 4.1, $A(\Phi'(\mathbb{Q}_p^n)) = \Phi'(\mathbb{Q}_p^n)$.

If A, B are pseudo-differential operators with symbols $\mathcal{A}(\xi), \mathcal{B}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$, respectively, then the operator AB is well defined and represented by the formula

$$(AB)f = F^{-1}[\mathcal{A}\mathcal{B}\, F[f]] \in \Phi'(\mathbb{Q}_p^n).$$

If $\mathcal{A}(\xi) \neq 0$, $\xi \in \mathbb{Q}_p^n \setminus \{0\}$ then we define the inverse pseudo-differential operator by the formula

$$A^{-1}f = F^{-1}[\mathcal{A}^{-1}\, F[f]], \quad f \in \Phi'(\mathbb{Q}_p^n).$$

Thus the family of pseudo-differential operators A with symbols $\mathcal{A}(\xi) \neq 0$, $\xi \in \mathbb{Q}_p^n \setminus \{0\}$ forms an Abelian group.

If the symbol $\mathcal{A}(\xi)$ of the operator A is a *homogeneous* or an *associated homogeneous* function (see Definition 2.1) then the pseudo-differential operator A is called *homogeneous* or *associated homogeneous*.

4.2. **The Taibleson fractional operator.** Let us consider a pseudo- differential operator D^{α}_x with the symbol $\mathcal{A}(\xi) = |\xi|_p^{\alpha}$. Thus, according to (4.1),

$$\big(D^{\alpha}_x \phi\big)(x) = F^{-1}\big[|\xi|_p^{\alpha} F[\phi](\xi)\big](x), \quad \phi \in \Phi(\mathbb{Q}_p^n). \tag{4.5}$$

This multi-dimensional Taibleson fractional operator was introduced in [28, §2], [29, III.4.] on the space of distributions $\mathcal{D}'(\mathbb{Q}_p^n)$ for $\alpha \in \mathbb{C}$, $\alpha \neq -n$.

In view of formulas (2.12), (2.16), (2.17), the relation (4.5) can be rewritten as a convolution

$$\big(D^{\alpha}_x \phi\big)(x) \stackrel{def}{=} \kappa_{-\alpha}(x) * \phi(x) = \langle \kappa_{-\alpha}(x), \phi(x-\xi)\rangle, \quad x \in \mathbb{Q}_p^n,$$

where $\phi \in \Phi(\mathbb{Q}_p^n)$, $\alpha \neq 0, -n$. Here the distribution from $\mathcal{D}'(\mathbb{Q}_p^n)$

$$\kappa_{\alpha}(x) = \frac{|x|_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, \quad \alpha \neq 0,\ n, \qquad x \in \mathbb{Q}_p^n, \tag{4.6}$$

is called the multidimensional *Riesz kernel* [28, §2], [29, III.4.], where the function $|x|_p$, $x \in \mathbb{Q}_p^n$ is given by (2.1). The Riesz kernel has a removable singularity at $\alpha = 0$ and according to [28, §2], [29, III.4.], [30, VIII.2], we obtain that $\langle \kappa_0(x), \varphi(x)\rangle \stackrel{def}{=} \lim_{\alpha\to 0}\langle \kappa_{\alpha}(x), \varphi(x)\rangle = \varphi(0)$, for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, i.e.,

$$\kappa_0(x) \stackrel{def}{=} \lim_{\alpha\to 0} \kappa_{\alpha}(x) = \delta(x). \tag{4.7}$$

Using (2.13), (2.17), (4.6), and taking into account (3.1) (for details, see [3], [4]), we define $\kappa_n(\cdot)$ as a distribution from the *Lizorkin space of distributions* $\Phi'(\mathbb{Q}_p^n)$:

$$\kappa_n(x) \stackrel{def}{=} \lim_{\alpha\to n} \kappa_\alpha(x) = -\frac{1-p^{-n}}{\log p}\log|x|_p. \tag{4.8}$$

With the help of (2.16), (4.7), (4.8), it is easy to see that

$$\kappa_\alpha(x) * \kappa_\beta(x) = \kappa_{\alpha+\beta}(x), \quad \alpha,\beta\in\mathbb{C}, \tag{4.9}$$

holds in the sense of the Lizorkin space $\Phi'(\mathbb{Q}_p^n)$.

In view of (4.7), (4.8), the multi-dimensional Taibleson operator on the Lizorkin space of test functions is defined for all $\alpha\in\mathbb{C}$ as

$$\big(D^\alpha_x\phi\big)(x) \stackrel{def}{=} \kappa_{-\alpha}(x) * \phi(x) = \langle\kappa_{-\alpha}(x), \phi(x-\xi)\rangle, \quad x\in\mathbb{Q}_p^n, \tag{4.10}$$

where $\phi\in\Phi(\mathbb{Q}_p^n)$.

If $\alpha\neq n$ then the Riesz kernel $\kappa_\alpha(x)$ is a *homogeneous* distribution of degree $\alpha-n$, and if $\alpha=n$ then the Riesz kernel is an *associated homogeneous* distribution of degree 0 and order 1 (see Definitions 2.1,(b),(d)). Thus the Taibleson fractional operator D^α_x, $\alpha\neq -n$ is a *homogeneous* pseudo-differential operator of degree α, and D^{-n}_x is an *associated homogeneous* pseudo-differential operator of degree $-n$ and order 1 with the symbol $\mathcal{A}(\xi)=P(|\xi|_p^{-n})$ (see (2.15)).

According to Lemma 4.1, the Lizorkin space $\Phi(\mathbb{Q}_p^n)$ is invariant under the Taibleson fractional operator D^α_x and $D^\alpha_x(\Phi(\mathbb{Q}_p^n))=\Phi(\mathbb{Q}_p^n)$ [3].

In view of (4.2), (4.3), $(D^\alpha_x)^T=D^\alpha_x$ and for $f\in\Phi'(\mathbb{Q}_p^n)$ we have

$$\langle D^\alpha_x f,\phi\rangle \stackrel{def}{=} \langle f, D^\alpha_x\phi\rangle, \quad \forall\,\phi\in\Phi(\mathbb{Q}_p^n). \tag{4.11}$$

It is clear that $D^\alpha_x(\Phi'(\mathbb{Q}_p^n))=\Phi'(\mathbb{Q}_p^n)$. Moreover, in view of (4.9), the family of operators D^α_x, $\alpha\in\mathbb{C}$ on the Lizorkin space forms an Abelian group: if $f\in\Phi'(\mathbb{Q}_p^n)$ then $D^\alpha_x D^\beta_x f = D^\beta_x D^\alpha_x f = D^{\alpha+\beta}_x f$, $D^\alpha_x D^{-\alpha}_x f = f$, $\alpha,\beta\in\mathbb{C}$.

5. p-Adic wavelets

5.1. **One-dimensional p-adic wavelets.** Let $n=1$. Consider the set

$$I_p=\{a=p^{-\gamma}\big(a_0+a_1p+\cdots+a_{\gamma-1}p^{\gamma-1}\big):$$
$$\gamma\in\mathbb{N};\ a_j=0,1,\dots,p-1;\ j=0,1,\dots,\gamma-1\}. \tag{5.1}$$

This set can be identified with the factor group $\mathbb{Q}_p/\mathbb{Z}_p$. Let

$$J_{p;m}=\{s=p^{-m}\big(s_0+s_1p+\cdots+s_{m-1}p^{m-1}\big):$$
$$s_j=0,1,\dots,p-1;\ j=0,1,\dots,m-1; s_0\neq 0\}, \tag{5.2}$$

where $m\geq 1$ is a *fixed* positive integer.

Let us introduce the function $\theta_s^{(m)}(x)=\chi_p(sx)\Omega\big(|x|_p\big)$, $x\in\mathbb{Q}_p$, $s\in J_{p;m}$, and the functions generated by its dilatations and translations:

$$\theta_{\gamma sa}^{(m)}(x)=p^{-\gamma/2}\chi_p\big(s(p^{\gamma}x-a)\big)\Omega\big(|p^{\gamma}x-a|_p\big),\quad x\in\mathbb{Q}_p, \tag{5.3}$$

where $\gamma\in\mathbb{Z}$, $s\in J_{p;m}$, $a\in I_p$, $\Omega(t)$ is the characteristic function (2.5) of the segment $[0,1]$.

Making the change of variables $\xi=p^{\gamma}x-a$ and taking into account (2.10), we obtain

$$\int_{\mathbb{Q}_p}\theta_{\gamma sa}^{(m)}(x)\,dx=p^{\gamma/2}\int_{\mathbb{Q}_p}\chi_p\big(s\xi\big)\Omega\big(|\xi|_p\big)\,d\xi=p^{\gamma/2}\Omega\big(|s|_p\big)=0. \tag{5.4}$$

Thus, in view of Theorem 5.1 (see below), one can see that the functions (5.3) are *p-adic wavelets*. Moreover, according to (5.4) and Lemma 3.1, the functions $\theta_{\gamma sa}^{(m)}(x)$ belong to the Lizorkin space $\Phi(\mathbb{Q}_p)$.

It is clear that for any $\gamma\in\mathbb{Z}$ and $s\in J_{p;m}$ the functions (5.4) are *periodical* with the periods $T_{\gamma s}\in p^{m-\gamma}\mathbb{Z}_p$.

In the case $m=1$, i.e., for $s=p^{-1}j$, $j=1,2,\dots,p-1$ these wavelets coincide with the Kozyrev wavelets [20]:

$$\theta_{\gamma sa}^{(1)}(x)=\theta_{\gamma ja}(x)=p^{-\gamma/2}\chi_p\big(p^{-1}j(p^{\gamma}x-a)\big)\Omega\big(|p^{\gamma}x-a|_p\big),\quad x\in\mathbb{Q}_p, \tag{5.5}$$

$\gamma\in\mathbb{Z}$, $j=1,2,\dots,p-1$, $a\in I_p$.

In particular, $\theta_s^{(1)}(x)=\theta_j(x)=\chi_p(p^{-1}jx)\Omega\big(|x|_p\big)$ for $j=1$. Since $|x|_p\le 1$, $x\in\mathbb{Q}_p$, i.e., $x=x_0+x_1p+x_2p^2+\cdots$, we have $p^{-1}x=p^{-1}x_0+x_1+x_2p+\cdots$, i.e., the fractional part (2.6) of a number $p^{-1}x$ is equal to $\{p^{-1}x\}_p=p^{-1}x_0$. According to (2.4),

$$\theta_1(x)=\chi_p(p^{-1}x)\Omega\big(|x|_p\big)=\begin{cases}0, & |x|_p\ge p,\\ e^{2\pi i\frac{r}{p}}, & x\in B_{-1}(r),\ r=1,\dots,p-1,\\ 1, & x\in B_{-1}.\end{cases} \tag{5.6}$$

Thus the function $\theta_1(x)=\chi_p(p^{-1}x)\Omega\big(|x|_p\big)$ takes values in the set $\{0,e^{2\pi i\frac{r}{p}}:r=0,1,\dots,p-1\}$ of $p+1$ elements.

Now we consider $\theta_s^{(1)}(x)=\theta_j(x)=\chi_p(p^{-1}jx)\Omega\big(|x|_p\big)$. Since $|jx|_p\le 1$, $x\in\mathbb{Q}_p$, we have $jx=y_0+y_1p+y_2p^2+\cdots$, $p^{-1}jx=p^{-1}y_0+y_1+y_2p+\cdots$, and $\{p^{-1}jx\}_p=p^{-1}y_0$ (see (2.6)). Thus,

$$\theta_j(x)=\chi_p(p^{-1}jx)\Omega\big(|x|_p\big)=\begin{cases}0, & |x|_p\ge p,\\ e^{2\pi i\{\frac{jr}{p}\}_p}, & x\in B_{-1}(r),\ r=1,\dots,p-1,\\ 1, & x\in B_{-1}.\end{cases}$$

It is clear that for the Kozyrev wavelets the *scaling function* is the characteristic function of the unit disc $\Delta_0(x)=\Omega\big(|x|_p\big)$, $x\in\mathbb{Q}_p$, and in view of (2.4) it satisfies the *two-scale equation*:

$$\Delta_0(x)=p^{-1/2}\sum_{r=0}^{p-1}h_r\Delta_0\Big(\frac{1}{p}x-\frac{r}{p}\Big),\quad x\in\mathbb{Q}_p, \tag{5.7}$$

where $h_r = p^{1/2}$. Relations (5.6), (5.7) imply that

$$\theta_1(x) = \chi_p(p^{-1}x)\Omega\big(|x|_p\big) = p^{-1/2}\sum_{r=0}^{p-1}\tilde{h}_r\Delta_0\Big(\frac{1}{p}x - \frac{r}{p}\Big), \quad x \in \mathbb{Q}_p, \tag{5.8}$$

where $\tilde{h}_r = p^{1/2}e^{2\pi i\frac{r}{p}}$, $r = 0, 1, \dots, p-1$. Similarly to (5.8), we have

$$\theta_j(x) = \chi_p(p^{-1}jx)\Omega\big(|x|_p\big) = p^{-1/2}\sum_{r=0}^{p-1}\tilde{h}_r\Delta_0\Big(\frac{1}{p}x - \frac{r}{p}\Big), \quad x \in \mathbb{Q}_p, \tag{5.9}$$

where $\tilde{h}_r = p^{1/2}e^{2\pi i\{\frac{jr}{p}\}_p}$, $r = 0, 1, \dots, p-1$.

In the same way we consider the function $\theta_s^{(m)}(x) = \chi_p(sx)\Omega\big(|x|_p\big)$, $s \in J_{p;m}$. Let $B_0 = \cup_a B_{-m}(a) \cup B_{-m}$ be the *canonical covering* (2.3) of the disc B_0 with p^m discs, $m \ge 1$, where $a = 0$ and $a = a_rp^r + a_{r+1}p^{r+1} + \cdots + a_{m-1}p^{m-1}$ is the center of the discs B_{-m} and $B_{-m}(a)$, respectively, $r = 0, 1, 2, \dots, m-1$, $0 \le a_j \le p-1$, $a_r \ne 0$.

For $x \in B_{-m}(a)$, $s \in J_{p;m}$, we have $x = a + p^m\big(y_0 + y_1p + y_2p^2 + \cdots\big)$, $s = p^{-m}\big(s_0 + s_1p + \cdots + s_{m-1}p^{m-1}\big)$, $s_0 \ne 0$; $sx = sa + \xi$, $\xi \in \mathbb{Z}_p$; and $\{sx\}_p = \{sa\}_p = \{p^{r-m}\big(a_r + a_{r+1}p + \cdots + a_{m-1}p^{m-r-1}\big)\big(s_0 + s_1p + \cdots + s_{m-1}p^{m-1}\big)\}_p$, $r = 0, 1, 2, \dots, m-1$, (see (2.6)). Thus,

$$\begin{aligned}\theta_s^{(m)}(x) &= \chi_p(sx)\Omega\big(|x|_p\big)\\ &= \begin{cases} 0, & |x|_p \ge p,\\ e^{2\pi i\{sa\}_p}, & x \in B_{-m}(a), \quad a = \sum_{l=r}^{m-1} a_lp^l,\\ 1, & x \in B_{-m},\end{cases}\end{aligned}$$

where $s = p^{-m}\big(s_0 + s_1p + \cdots + s_{m-1}p^{m-1}\big)$, $0 \le s_j \le p-1$, $j = 0, 1, \dots, m-1$, $s_0 \ne 0$; $a = a_rp^r + a_{r+1}p^{r+1} + \cdots + a_{m-1}p^{m-1}$, $0 \le a_j \le p-1$, $a_r \ne 0$, $r = 0, 1, \dots, m-1$. Thus the function $\theta_s^{(m)}(x) = \chi_p(sx)\Omega\big(|x|_p\big)$ takes values in the set $\{0, 1, e^{2\pi i\{sa\}_p}\}$ of $p^m + 1$ elements.

In this case, using the *scaling function*, we obtain

$$\theta_s^{(m)}(x) = \chi_p(sx)\Omega\big(|x|_p\big) = p^{-m/2}\sum_a \tilde{h}_a\Delta_0\Big(\frac{1}{p^m}x - \frac{a}{p^m}\Big), \tag{5.10}$$

$x \in \mathbb{Q}_p$, where $\tilde{h}_0 = p^{m/2}$; $\tilde{h}_a = p^{m/2}e^{2\pi i\{sa\}_p}$, $a = a_rp^r + a_{r+1}p^{r+1} + \cdots + a_{m-1}p^{m-1}$, $r = 0, 1, \dots, m-1$, $0 \le a_j \le p-1$, $a_r \ne 0$.

Theorem 5.1. *The functions* (5.3) *form an orthonormal complete basis in* $\mathcal{L}^2(\mathbb{Q}_p)$ *(p-adic wavelet basis).*

Proof. Consider the scalar product

$$\begin{aligned}&\big(\theta_{\gamma' s' a'}^{(m)}(x), \theta_{\gamma s a}^{(m)}(x)\big) = p^{-(\gamma+\gamma')/2}\\ &\times \int_{\mathbb{Q}_p}\chi_p\big(s'(p^{\gamma'}x - a') - s(p^{\gamma}x - a)\big)\Omega\big(|p^{\gamma}x - a|_p\big)\Omega\big(|p^{\gamma'}x - a'|_p\big)\,dx.\end{aligned} \tag{5.11}$$

If $\gamma \le \gamma'$, according to formula [30, VII.1], [20]

$$\Omega\big(|p^{\gamma}x - a|_p\big)\Omega\big(|p^{\gamma'}x - a'|_p\big) = \Omega\big(|p^{\gamma}x - a|_p\big)\Omega\big(|p^{\gamma'-\gamma}a - a'|_p\big), \tag{5.12}$$

(5.11) can be rewritten as

$$\begin{aligned}\big(\theta^{(m)}_{\gamma' s' a'}(x), \theta^{(m)}_{\gamma s a}(x)\big) &= p^{-(\gamma+\gamma')/2}\Omega\big(|p^{\gamma'-\gamma}a - a'|_p\big) \\ &\times \int_{\mathbb{Q}_p} \chi_p\big(s'(p^{\gamma'}x - a') - s(p^{\gamma}x - a)\big)\Omega\big(|p^{\gamma}x - a|_p\big)\,dx.\end{aligned} \tag{5.13}$$

Let $\gamma < \gamma'$. Making the change of variables $\xi = p^{\gamma}x - a$ and taking into account (2.10), from (5.13) we obtain

$$\begin{aligned}\big(\theta^{(m)}_{\gamma' s' a'}(x), \theta^{(m)}_{\gamma s a}(x)\big) &= p^{-(\gamma+\gamma')/2}\chi_p\big(s'(p^{\gamma'-\gamma}a - a')\big) \\ &\times\Omega\big(|p^{\gamma'-\gamma}a - a'|_p\big)\int_{\mathbb{Q}_p} \chi_p\big((p^{\gamma'-\gamma}s' - s)\xi\big)\Omega\big(|\xi|_p\big)\,d\xi \\ &= p^{-(\gamma+\gamma')/2}\chi_p\big(s'(p^{\gamma'-\gamma}a - a')\big)\Omega\big(|p^{\gamma'-\gamma}a - a'|_p\big)\Omega\big(|p^{\gamma'-\gamma}s' - s|_p\big).\end{aligned} \tag{5.14}$$

Since

$$\begin{aligned} p^{\gamma'-\gamma}s' &= p^{\gamma'-\gamma-m}\big(s'_0 + s'_1 p + \cdots + s'_{\gamma-1}p^{m-1}\big), \\ s &= p^{-m}\big(s_0 + s_1 p + \cdots + s_{\gamma-1}p^{m-1}\big),\end{aligned}$$

where $s'_0, s_0 \ne 0$, $\gamma' - \gamma \le 1$, it is clear that fractional part $\{p^{\gamma'-\gamma}s' - s\}_p \ne 0$. Thus $\Omega\big(|p^{\gamma'-\gamma}s' - s|_p\big) = 0$ and $\big(\theta^{(m)}_{\gamma' s' a'}(x), \theta^{(m)}_{\gamma s a}(x)\big) = 0$.

Consequently, the scalar product $\big(\theta^{(m)}_{\gamma' s' a'}(x), \theta^{(m)}_{\gamma s a}(x)\big) = 0$ can be nonzero only if $\gamma = \gamma'$. In this case (5.14) implies

$$\big(\theta^{(m)}_{\gamma s' a'}(x), \theta^{(m)}_{\gamma s a}(x)\big) = p^{-\gamma}\chi_p\big(s'(a - a')\big)\Omega\big(|a - a'|_p\big)\Omega\big(|s' - s|_p\big), \tag{5.15}$$

where $\Omega\big(|a - a'|_p\big) = \delta_{a'a}$, $\Omega\big(|s' - s|_p\big) = \delta_{s's}$, and $\delta_{s's}$, $\delta_{a'a}$ are the Kronecker symbols.

Since $\int_{\mathbb{Q}_p}\Omega\big(|p^{\gamma}x - a|_p\big)\,dx = p^{\gamma}$ [30, IV,(2.3)], formulas (5.14), (5.15) imply that

$$\big(\theta^{(m)}_{\gamma' s' a'}(x), \theta^{(m)}_{\gamma s a}(x)\big) = \delta_{\gamma'\gamma}\delta_{s's}\delta_{a'a}. \tag{5.16}$$

Thus the system of functions (5.3) is orthonormal.

To prove the completeness of the system of functions (5.3), we repeat the corresponding proof [20] almost word for word. Recall that the system of the characteristic functions of the discs B_k is complete in $\mathcal{L}^2(\mathbb{Q}_p)$. Consequently, taking into account that the system of functions $\{\theta^{(m)}_{\gamma s a}(x) : \gamma \in \mathbb{Z}, s \in J_{p;m}, a \in I_p\}$ is invariant under dilatations and translations, in order to prove that it is a complete system, it is sufficient to verify the Parseval identity for the characteristic function $\Omega(|x|_p)$.

If $0 \le \gamma$, according to (5.12), (2.10),

$$\big(\Omega(|x|_p), \theta^{(m)}_{\gamma s a}(x)\big) = p^{-\gamma/2}\Omega\big(|-a|_p\big)\int_{\mathbb{Q}_p}\chi_p\big(s(p^{\gamma}x - a)\big)\Omega\big(|x|_p\big)\,dx$$

$$= p^{-\gamma/2}\chi_p\big(-sp^{\gamma}a\big)\big)\Omega\big(|sp^{\gamma}|_p\big)\Omega\big(|-a|_p\big) = \begin{cases} 0, & a \neq 0, \\ 0, & a = 0,\ \gamma \le m-1, \\ p^{-\gamma/2}, & a = 0,\ \gamma \ge m. \end{cases}$$

If $0 > \gamma$, according to (5.12), (2.10),

$$\big(\Omega(|x|_p), \theta^{(m)}_{\gamma s a}(x)\big) = p^{-\gamma/2}\Omega\big(|p^{-\gamma}a|_p\big)\int_{\mathbb{Q}_p}\chi_p\big(s(p^{\gamma}x-a)\big)\Omega\big(|p^{\gamma}x-a|_p\big)\,dx$$

$$= p^{-\gamma/2}\Omega\big(|p^{-\gamma}a|_p\big)\int_{\mathbb{Q}_p}\chi_p\big(s\xi\big)\Omega\big(|\xi|_p\big)\,d\xi = p^{-\gamma/2}\Omega\big(|p^{-\gamma}a|_p\big)\Omega\big(|s|_p\big) = 0.$$

Thus,

$$\sum_{\gamma\in\mathbb{Z}, s\in J_{p;m}, a\in I_p}\big|\big(\Omega(|x|_p), \theta^{(m)}_{\gamma s a}(x)\big)\big|^2 = \sum_{\gamma=m}^{\infty}\sum_{s\in J_{p;m}} p^{-\gamma}$$

$$= p^{m-1}(p-1)\frac{p^{-m}}{1-p^{-1}} = 1 = \big|\big(\Omega(|x|_p), \Omega(|x|_p)\big)\big|^2.$$

Thus the system of functions (5.3) is an orthonormal basis in $\mathcal{L}^2(\mathbb{Q}_p)$ (p-adic wavelet basis). □

Corollary 5.1. *The functions*

$$\widetilde{\theta}^{(m)}_{\gamma s a} = F[\theta^{(m)}_{\gamma s a}](\xi) = p^{\gamma/2}\chi_p\big(p^{-\gamma}a\cdot\xi\big)\Omega\big(|s+p^{-\gamma}\xi|_p\big), \quad \xi\in\mathbb{Q}_p,$$

form an orthonormal complete basis in $\mathcal{L}^2(\mathbb{Q}_p)$, $a\in I_p$; $s\in J_{p;m}$; $m\ge 1$ is a fixed positive integer.

The proof follows from Theorem 5.1, formula (6.3) (see below) and the Parseval formula [30, VII,(4.1)]

5.2. **Multidimensional p-adic wavelets.** Let us introduce n-dimensional functions generated by the n-direct product of the one-dimensional p-adic wavelets (5.3):

$$\Theta^{(m)}_{\gamma s a}(x) = p^{-n\gamma/2}\chi_p\big(s\cdot(p^{\gamma}x-a)\big)\Omega\big(|p^{\gamma}x-a|_p\big), \tag{5.17}$$

$x=(x_1,\dots,x_n)\in\mathbb{Q}_p^n$, where $\gamma\in\mathbb{Z}$; $a=(a_1,\dots,a_n)\in I_p^n$; $s=(s_1,\dots,s_n)\in J_{p;m}^n$; $m=(m_1,\dots,m_n)$, $m_j\ge 1$ is a *fixed* positive integer, $j=1,2,\dots,n$. Here $I_p^n = I_p\times\cdots\times I_p$ and $J_{p;m}^n = J_{p;m_1}\times\cdots\times J_{p;m_n}$ are the n-direct products of the corresponding sets (5.1) and (5.2).

Using (5.4), (2.2), it is easy to verify that

$$\int_{\mathbb{Q}_p}\Theta^{(m)}_{\gamma s a}(x)\,d^nx = 0. \tag{5.18}$$

Thus the functions (5.17) are *n-dimensional p-adic wavelets*. According to (5.18) and Lemma 3.1, $\Theta^{(m)}_{\gamma s a}(x)$ belong to the Lizorkin space $\in\Phi(\mathbb{Q}_p^n)$.

For any $\gamma\in\mathbb{Z}$ and $s=(s_1,\dots,s_n)\in J_{p;m}^n$ the functions (5.17) are periodical with the vector periods $T_{\gamma s}=(T_{1|\gamma s},\dots,T_{n|\gamma s})\in p^{m-\gamma}\mathbb{Z}_p^n$.

In view of (2.2), Theorem 5.1 implies the following statement.

Theorem 5.2. *The functions* (5.17) *form an orthonormal complete basis in* $\mathcal{L}^2(\mathbb{Q}_p^n)$ *(p-adic wavelet basis).*

Corollary 5.2. *The functions*

$$\widetilde{\Theta}_{\gamma sa}^{(m)} = F[\Theta_{\gamma sa}^{(m)}](\xi) = p^{n\gamma/2}\chi_p\big(p^{-\gamma}a\cdot\xi\big)\Omega\big(|s+p^{-\gamma}\xi|_p\big), \quad \xi\in\mathbb{Q}_p^n,$$

form an orthonormal complete basis in $\mathcal{L}^2(\mathbb{Q}_p^n)$, $a=(a_1,\dots,a_n)\in I_p^n$; $s=(s_1,\dots,s_n)\in J_{p;m}^n$; $m=(m_1,\dots,m_n)$, $m_j\ge 1$ *is a fixed positive integer,* $j=1,2,\dots,n$.

The proof follows from Theorem 5.2, formula (6.3) (see below) and the Parseval formula [30, VII,(4.1)].

6. p-Adic wavelets as eigenfunctions of pseudo-differential operators

6.1. **Pseudo-differential operators.** As mentioned above, the one- dimensional Kozyrev wavelets (5.5) introduced in [20] is a particular case of the wavelets (5.3) for $m=1$. Moreover, in [20] S. V. Kozyrev proved that his wavelets (5.5) are eigenfunctions of the one-dimensional Vladimirov operator D^α for $\alpha>0$:

$$D^\alpha\theta_{\gamma ja}(x)=p^{\alpha(1-\gamma)}\theta_{\gamma ja}(x), \quad x\in\mathbb{Q}_p,$$

where $\gamma\in\mathbb{Z}$, $a\in I_p$, $j=1,2,\dots p-1$. Later, it was proved in [3, 4.4.] that in fact, the Kozyrev wavelets (5.5) are eigenfunctions of the Vladimirov operator for any α, i.e., the above formula holds for all $\alpha\in\mathbb{C}$.

Now we prove that n-dimensional wavelets (5.17) are eigenfunctions for a class of pseudo-differential operators (4.1), which includes the Taibleson fractional operator (4.10), (4.6).

Theorem 6.1. *Let A be a pseudo-differential operator with a symbol $\mathcal{A}(\xi)\in\mathcal{E}(\mathbb{Q}_p^n\setminus\{0\})$. Then the n-dimensional p-adic wavelet* (5.17) *is an eigenfunction of A if and only if*

$$\mathcal{A}\big(p^\gamma(-s+\eta)\big)=\mathcal{A}\big(-p^\gamma s\big), \qquad \forall\,\eta\in\mathbb{Z}_p^n, \tag{6.1}$$

where $\gamma\in\mathbb{Z}$, $j\in J_{p;m}^n$, $a\in I_p^n$. Here the eigenvalue $\lambda=\mathcal{A}\big(-p^\gamma s\big)$, i.e.,

$$A\Theta_{\gamma sa}^{(m)}(x)=\mathcal{A}(-p^\gamma s)\Theta_{\gamma sa}^{(m)}(x).$$

Proof. Let $\Theta_s^{(m)}(x)=\chi_p(s\cdot x)\Omega(|x|_p)$; $x\in\mathbb{Q}_p^n$; $s=(s_1,\dots,s_n)\in J_{p;m}^n$, $s_k\in J_{p;m_k}$, $k=1,2,\dots,n$. Using (2.2), (2.10), (2.9), we have

$$F[\Theta_s^{(m)}(x)](\xi)=F\Big[\prod_{k=1}^n\chi_p(x_ks_k)\Omega(|x_k|_p)\Big](\xi)=\prod_{k=1}^n F\Big[\Omega(|x_k|_p)\Big](\xi_k+s_k|_p)$$

$$=\prod_{k=1}^n\Omega\big(|\xi_k+s_k|_p\big)=\Omega\big(|\xi+s|_p\big), \quad \xi\in\mathbb{Q}_p^n. \tag{6.2}$$

Here, in view of (2.2), $\Omega\big(|\xi+s|_p\big)=\Omega\big(|\xi_1+s_1|_p\big)\times\cdots\times\Omega\big(|\xi_n+s_n|_p\big)$.

According to (5.2), $|s_k|_p = p^{m_k}$, i.e., $\Omega\big(|\xi_k + s_k|_p\big) \neq 0$ only if $\xi_k = -s_k + \eta_k$, where $\eta_k \in \mathbb{Z}_p$, $s_k \in J_{p;m_k}$, $k = 1, 2, \dots, n$. Thus $\xi = -s + \eta$, where $\eta \in \mathbb{Z}_p^n$, $s \in J_{p;m}^n$, and in view of (2.1), $|\xi|_p = p^{\max\{m_1,\dots,m_n\}}$.

In view of formulas (5.17), (6.2), (2.9), we have

$$
\begin{aligned}
F[\Theta_{\gamma s a}^{(m)}(x)](\xi) &= p^{-n\gamma/2} F[\Theta_s^{(m)}(p^\gamma x - a)](\xi) \\
&= p^{n\gamma/2} \chi_p\big(p^{-\gamma} a \cdot \xi\big) \Omega\big(|s + p^{-\gamma}\xi|_p\big). \qquad (6.3)
\end{aligned}
$$

Let condition (6.1) be satisfied. Then (4.1), (6.3) imply

$$
\begin{aligned}
A\Theta_{\gamma s a}^{(m)}(x) &= F^{-1}\big[\mathcal{A}(\xi) F[\Theta_{\gamma s a}^{(m)}](\xi)\big](x) \\
&= p^{n\gamma/2} F^{-1}\big[\mathcal{A}(\xi)\chi_p\big(p^{-\gamma} a \cdot \xi\big)\Omega\big(|s + p^{-\gamma}\xi|_p\big)\big](x). \qquad (6.4)
\end{aligned}
$$

Making the change of variables $\xi = p^\gamma(\eta - s)$ and using (2.10), we obtain

$$
\begin{aligned}
A\Theta_{\gamma s a}^{(m)}(x) &= p^{-n\gamma/2} \int_{\mathbb{Q}_p^n} \chi_p\big(-(p^\gamma x - a)\cdot(\eta - s)\big) \mathcal{A}(p^\gamma(\eta - s))\, \Omega(|\eta|_p)\, d^n\eta \\
&= p^{-n\gamma/2} \mathcal{A}(-p^\gamma s) \chi_p\big(s \cdot (p^\gamma x - a)\big) \int_{B_0^n} \chi_p(-(p^\gamma x - a)\cdot\eta)\, d^n\eta \\
&= \mathcal{A}(-p^\gamma s)\Theta_{\gamma s a}^{(m)}(x).
\end{aligned}
$$

Consequently, $A\Theta_{\gamma s a}^{(m)}(x) = \lambda\Theta_{\gamma s a}^{(m)}(x)$, where $\lambda = \mathcal{A}(-p^\gamma s)$.

Conversely, if $A\Theta_{\gamma s a}^{(m)}(x) = \lambda\Theta_{\gamma s a}^{(m)}(x)$, $\lambda \in \mathbb{C}$, then, using (4.1), (6.3), (6.4), we have

$$
\big(\mathcal{A}(\xi) - \lambda\big)\Omega\big(|s + p^{-\gamma}\xi|_p\big) = 0, \quad \xi \in \mathbb{Q}_p^n.
$$

The latter equation has a nontrivial solution only if $s + p^{-\gamma}\xi = \eta$, $\eta \in \mathbb{Z}_p^n$, i.e., $\xi = p^\gamma(-s + \eta)$ and $\lambda = \mathcal{A}\big(p^\gamma(-s + \eta)\big)$ for any $\eta \in \mathbb{Z}_p^n$. Thus $\lambda = \mathcal{A}(-p^\gamma s)$, and, consequently, (6.1) holds.

The proof of the theorem is complete. □

The following particular statement was proved in [3].

Corollary 6.1. ([3]) *Let A be a homogeneous pseudo-differential operator with a symbol $\mathcal{A}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$ of degree π_β. Then the n-direct product $\Theta_{\gamma j a}(x)$ of the one-dimensional Kozyrev p-adic wavelets* (5.5)*, i.e., the n-dimensional p-adic wavelet* (5.17) *$\Theta_{\gamma s a}^{(1)}$ (for $m = 1$) is an eigenfunction of A if and only if*

$$
\mathcal{A}(-p^{-1}j + \eta) = \mathcal{A}(-p^{-1}j), \qquad \forall\, \eta \in \mathbb{Z}_p^n, \qquad (6.5)
$$

where $\gamma \in \mathbb{Z}$; $a \in I_p^n$; $j = (j_1, \dots, j_n)$, $j_k = 1, 2, \dots, p-1$, $k = 1, 2, \dots, n$. Here the eigenvalue $\lambda = p^{(1-\beta)\gamma}\mathcal{A}(-p^{-1}j)$, i.e.,

$$
A\Theta_{\gamma j a}(x) = p^{(1-\beta)\gamma}\mathcal{A}(-p^{-1}j)\Theta_{\gamma j a}(x).
$$

6.2. **The Taibleson fractional operator.** As mentioned above, the Taibleson fractional operator D^{β}_{x} is homogeneous of degree β (see Definition 2.1) and has a symbol $\mathcal{A}(\xi)=|\xi|^{\beta}_{p}$, which satisfies the condition (6.1)

$$\mathcal{A}\big(p^{\gamma}(-s+\eta)\big)=|p^{\gamma}(-s+\eta)|^{\beta}_{p}=p^{-\beta\gamma}|-s|^{\beta}_{p}=p^{\beta(\max\{m_1,\dots,m_n\}-\gamma)}=\mathcal{A}\big(-p^{\gamma}s\big)$$

for all $\eta\in\mathbb{Z}^{n}_{p}$. Thus according to Theorem 6.1, the n-dimensional p-adic wavelet (5.17) is an eigenfunction of D^{β}_{x}:

$$D^{\beta}_{x}\Theta^{(m)}_{\gamma sa}(x)=p^{\beta(\max\{m_1,\dots,m_n\}-\gamma)}\Theta^{(m)}_{\gamma sa}(x),\quad \beta\in\mathbb{C},\quad x\in\mathbb{Q}^{n}_{p}, \tag{6.6}$$

$\gamma\in\mathbb{Z}$, $a\in I^{n}_{p}$, $s\in J^{n}_{p;m}$.

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